富山短期大学紀要第三十六巻

An Exact Penalty Parameter of an Inner Approximation Method for a Reverse Convex Programming Problem

YAMADA Syuuji, TANINO Tetsuzo and INUIGUCHI Masahiro

Abstract

In this paper, we consider a reverse convex programming problem constrained by a convex set and a reverse convex set which is defined by the complement of the interior of a compact convex set X. When X is not necessarily a polytope in the problem, an inner approximation method using penalty functions has been proposed by Yamada, Tanino and Inuiguchi [9]. In this paper, we show that there exists an exact penalty parameter of the proposed algorithm.

Keywords: Global Optimization, Reverse Convex Programming Problem, Inner Approximation Method, Penalty Function Method, Exact Penalty Parameter

1. Introduction

In this paper, we consider a reverse convex programming problem constrained by a convex set and a reverse convex set which is defined by the complement of the interior of a compact convex set X. When X is a polytope in the problem, a solution method using duality has been proposed (Horst and Tuy [4], Horst and Pardalos [5], Konno, Thach and Tuy [6], Tuy [8]). Duality is one of the most powerful tools in dealing with a global optimization problem like the problem described above. The dual problem to the problem is a quasi-convex maximization problem over a convex set and solving one of the original

やまだ　しゅうじ（経営情報学科）　たにの　てつぞう（大阪大学大学院工学研究科）
いぬいぐち　まさひろ（大阪大学大学院工学研究科）

and the dual problems is equivalent to solving the other (Konno, Thach and Tuy [6], Tuy [8]). Since the feasible set of the dual problem is a polytope, there exists a vertex which solves the dual problem. Moreover, since the objective function of the dual problem is the quasi-conjugate function of the objective function of the original problem, for every vertex, the objective function value is obtained by solving a constrained convex minimization problem. Consequently, an optimal solution of the original problem is obtained by solving a finite number of constrained convex minimization problems. When X is not necessarily a polytope, an inner approximation method incorporating with a penalty function method has been proposed by Yamada, Tanino and Inuiguchi [9]. The proposed algorithm utilizes inner approximation of X by a sequence of polytopes to generate relaxed problems.

In this paper, we show that there exists an exact penalty parameter of the proposed inner approximation algorithm. It follows from the existence of an exact penalty parameter that for a sufficiently large initial penalty parameter, an optimal solution of the reverse convex programming problem can be obtained by executing the inner approximation algorithm using penalty functions without replacing a penalty parameter.

The organization of this paper is as follows: In Section 2, we explain a reverse convex programming problem. Moreover, we describe an equivalent problem to the problem, and its dual problem, where equivalence is understood in the sense that the sets of optimal solutions coincide. In Section 3, we explain an inner approximation algorithm proposed by Yamada, Tanino and Inuiguchi [9]. In Section 4, we explain another inner approximation algorithm incorporating with a penalty function method proposed by Yamada, Tanino and Inuiguchi [9]. In Section 5, we show that there exists an exact penalty parameter of the algorithm explained in Section 4.

Throughout this paper, we use the following notation: int X, bd X and co X denote the interior set of $X \subset R^n$, the boundary set of X and the convex hull of X, respectively. Let $\bar{R}=R\cup\{-\infty\}\cup\{+\infty\}$. Let for $a,\ b\in R^n$, $]a,\ b[=\{x\in R^n:\ x=a+\delta\,(b-a),\ 0<\delta<1,\ \delta\in R\}$ and $]a,\ b]=\{x\in R^n:\ x=a+\delta\,(b-a),\ 0<\delta\leq 1,\ \delta\in R\}$. Given a convex polyhedral set (or polytope) $X\subset R^n$, $V(X)$ denotes the set of all vertices of X. For a subset $X\subset R^n$, $X^0=\{u\in R^n:\ \langle u,\ x\rangle\leq 1,\ \forall x\in X\}$ is called the polar set of X. For a nonempty closed set $X\subset R^n$, $N_X(y)$ denote the normal cone to X at $y\in X$. For a subset $X\subset R^n$, the indicator of X which is denoted by $\delta(\cdot|X)$ is an extended-real-valued function defined as follows:

$$\delta\,(x|X)\ \begin{cases} 0 & \text{if } x\in X. \\ +\infty & \text{if } x\notin X. \end{cases}$$

Given a function $f: R^n \rightarrow R\cup\{+\infty\}$, the quasi-conjugate of f^H is the function f^H defined as follows:

$$f^H(\mathrm{u}) \begin{cases} -\sup\{f(x) : x \in R^n\} & \text{if } u=0 \\ -\inf\{f(x) : \langle u, x\rangle \geq 1\} & \text{if } u \neq 0. \end{cases}$$

The gradient of f at x is denoted by $\nabla f(x)$ and the subdifferential of f at x by $\partial f(x)$.

2 A Reverse Convex Programming Problem

Let us consider the following reverse convex programming problem problem:

$$(RCP) \begin{cases} \text{minimize} & f(x), \\ \text{subject to} & x \in Y \setminus \operatorname{int} X, \end{cases}$$

where $f: R^n \to R$ is a convex function, X is a compact convex set and Y is a closed convex set in R^n. In general, the feasible set of problem (RCP) is not convex. For problem (RCP), we shall assume the following throughout this paper:

(A1) $Y \setminus \operatorname{int} X \neq \emptyset$.

(A2) For some $\alpha \in R$, $\{x \in R^n : f(x) \leq \alpha\}$ is nonempty and compact.

(A3) $X = \{x \in R^n : p_j(x) \leq 0,\ j=1,\dots,t_X\}$ and $Y = \{x \in R^n : r_j(x) \leq 0,\ j=1,\dots,t_Y\}$ where $p_j : R^n \to R$ ($j=1,\dots,t_X$) and $r_j : R^n \to R$ ($j=1,\dots,t_Y$) are convex functions. Moreover, there exists $x_X, x_Y \in R^n$ such that $p_j(x_X) < 0$ ($j=1,\dots,t_X$) and $r_j(x_Y) < 0$ ($j=1,\dots,t_Y$).

Let $p(x) = \max_{j=1,\dots,t_X} p_j(x)$ and $r(x) = \max_{j=1,\dots,t_Y} r_j(x)$. Then, from assumption (A3), $X = \{x \in R^n : p(x) \leq 0\}$, $Y = \{x \in R^n : r(x) \leq 0\}$, $\operatorname{int} X = \{x \in R^n : p(x) \leq 0\}$ and $\operatorname{int} Y = \{x \in R^n : r(x) \leq 0\}$. From assumption (A2), the minimal value of f over R^n exists. Moreover, for any $\beta \geq \min\{f(x) : x \in R^n\}$, $\{x \in R^n : f(x) \leq \beta\}$ is nonempty and compact. From assumption (A1), there exists a feasible solution x' of problem (RCP). Then, problem (RCP) is equivalent to minimize $f(x)$ subject to $x \in (Y \setminus \operatorname{int} X) \cap \{x \in R^n : f(x) \leq f(x')\}$. Since $\{x \in R^n : f(x) \leq f(x')\}$ is compact, problem (RCP) has an optimal solution. Denote by $\min(RCP)$ the optimal value of problem (RCP). Then, we have $\min(RCP) < +\infty$. From assumptions (A1) and (A2), Y is nonempty and there exists a minimal solution x^0 of f over Y. Then, it is fairly easy to find x^0. In case $x^0 \in R^n \setminus \operatorname{int} X$, x^0 solves problem (RCP). In the other case, we propose a solution method in this paper. Throughout this paper, without loss of generality, we may assume the following:

(A4) $p(0) < 0$ and $r(0) \leq 0$, that is, $0 \in \operatorname{int} X$ and $0 \in Y$. Moreover, $0 \in R^n$ is a minimal solution of f over Y.

(A5) For any $x \in (\operatorname{bd} X) \cap Y$ and $w \in \partial p(x)$, $\{y \in R^n : \langle w, y-x\rangle \geq 0\} \cap \operatorname{int} Y \neq 0$.

By using the indicator of Y, problem (RCP) can be reformulated as

$$(MP)\begin{cases}\text{minimize} & g(x)\\ \text{subject to} & x\in R^n\setminus\text{int }X\end{cases}$$

where $g(x) := f(x) + \delta(x|Y)$. The objective function $g : R^n \to \bar{R}$ is a quasi-convex function. From assumption (A4), we have $g(0)=\inf\{g(x) : x\in R^n\}$. The dual problem of problem (MP) is formulated as

$$(DP)\begin{cases}\text{maximize} & g^H(u)\\ \text{subject to} & u\in X^0.\end{cases}$$

Hence, by assumption (A4) and the principle of the duality, X^0 is a compact convex set. Furthermore, since g^H is a quasi-convex function (Konno, Thach and Tuy[6], Chapter 2), we note that problem (DP) is a quasi-convex maximization problem over a compact convex set in R^n. Denote by min (MP) and max (DP) the optimal values of (MP) and (DP), respectively. Since problem (MP) is equivalent to problem (RCP), we have min $(MP)=$min $(RCP)<+\infty$. Moreover, it follows from the duality relation between problems (MP) and (DP) that min $(MP)=-$max (DP) (cf., Konno, Thach and Tuy[6], Chapter 4).

3 An Inner Approximation Method for Problem (MP)

3.1 Relaxed Problems for Problems (MP) and (DP)

One of the reasons for difficulty in solving problem (MP) is that X is not a polytope. If X is a polytope, then the feasible set of problem (MP) can be formulated as the union of finite halfspaces. In this case, problem (MP) is fairly easy to solve by minimizing g over every halfspace.

In this subsection, we discuss the following problem:

$$(P)\begin{cases}\text{minimize} & g(x),\\ \text{subject to} & x\in R^n\setminus\text{int }S,\end{cases}$$

where S is a polytope such that $S\subset X$ and $0\in\text{int }S$. Then, we get $R^n\setminus\text{int }S\supset R^n\setminus\text{int }X$. Therefore, problem (P) is a relaxed problem for problem (MP). From the definition of g, we note that problem (P) is equivalent to minimize $f(x)$ subject to $x\in Y\setminus\text{int }S$. Since $(Y\setminus\text{int }S)\supset(Y\setminus\text{int }X)\neq 0$, by assumption (A2), a minimal solution of f on $Y\setminus\text{int }S$ exists and solves problem (P). Denote by min (P) the optimal value of problem (P). Then, we have min $(P)\leq$min $(MP)<+\infty$

The dual problem of problem (P) is formulated as

$$(D)\quad \begin{cases} \text{maximize} & g^H(u), \\ \text{subject to} & u \in S^0. \end{cases}$$

Since $S \subset X$, the feasible set of problem (D) includes X^0. Therefore, problem (D) is a relaxed problem of (DP). We note that the feasible set S^0 is a polytope because S is a polytope and $0 \in \text{int } S$. Hence, problem (D) is a quasi-convex maximization over a polytope S^0. There exists an optimal solution of problem (D) over the set of all vertices of S^0. Denote by max (D) the optimal value of problem (D). Since problem (D) is the dual problem of problem (P) and a relaxed problem of problem (DP), we obtain max $(D) = -\min(P) \geq -\min(MP) = \max(DP) > -\infty$ (Konno, Thach and Tuy[6], Chapter 4). Consequently, we can choose an optimal solution of problem (D) from $V(S^0)$. Since $0 \in \text{int } S$, from the principle of duality, we have

$$S^0 = \{u \in R^n : \langle u, z\rangle \leq 1, \ \forall z \in V(S)\} \text{ and } S = \{x \in R^n : \langle u, x\rangle \leq 1, \ \forall v \in V(S^0)\}.$$

Hence, we obtain $0 \in V(S^0)$.

For any $v \in V(S^0)$, we have $g^H(v) = -\inf\{g(x) : \langle v, x\rangle \geq 1\}$. From the definition of g, for any $v \in V(S)$,

$$g^H(v) \quad \begin{cases} -\infty, & \text{if } Y \cap \{x \in R^n : \langle v, x\rangle \geq 1\} = \emptyset, \\ -\inf\{f(x) : \langle v, x\rangle \geq 1, \ x \in Y\} & \text{otherwise.} \end{cases}$$

This implies that $v \in V(S^0)$ is not optimal to problem(D) if $Y \cap \{x \in R^n : \langle v, x\rangle \geq 1\} = \emptyset$.

Lemma 3.1 [9] *There exists $v \in V(S^0)$ such that $Y \cap \{x \in R^n : \langle v, x\rangle \geq 1\} \neq \emptyset$.*

Denote by Γ the set of all $v \in V(S^0)$ such that $Y \cap \{x \in R^n : \langle v, x\rangle \geq 1\} \neq \emptyset$. From Lemma 3.1, $\Gamma \neq \emptyset$. For every $v \in \Gamma$, we consider the following convex minimization problem:

$$(SP(v))\quad \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in Y \cap \{x \in R^n : \langle v, x\rangle \geq 1\}. \end{cases}$$

From assumption (A2), for every $v \in \Gamma$, problem $(SP(v))$ has an optimal solution x^v. Then, we have $g^H(v) = -\min(SP(v)) = -f(x^v)$, where min $(SP(v))$ is the optimal value of problem $(SP(v))$. Hence, $\hat{v} \in \Gamma$ is an optimal solution of problem(D) if $f(x^{\hat{v}}) = \min\{f(x^v) : v \in V(S^0)\}$. Moreover, $x^{\hat{v}}$ is optimal to problem (P) (Konno, Thach and Tuy[7], Proposition 4.3). However, it is hard to examine whether $Y \cap \{x \in R^n : \langle v, x\rangle \geq 1\}$ is empty. This examination is not necessary to execute the inner approximation algorithm proposed in Section 4.

3.2 An Inner Approximation Algorithm

From the discussion in Subsection 3.1, we notice that inner approximation of X by a sequence of polytopes is applicable in solving problem (MP).

The inner approximation algorithm proposed by Yamada, Tanino and Inuiguchi [9] is as follows:

Algorithm IA

Initialization. Generate a finite set V_1 such that $V_1 \subset X$ and that $0 \in \mathrm{int}$ (co V_1). Let S_1 $=$co V_1. Compute the vertex set $V((S_1)^\circ)$. For convenience, let $V((S_0)^\circ)=\emptyset$. Set $k \leftarrow 1$ and go to Step 1.

Step 1. Let Γ_k be the set of all $v \in V((S_k)^\circ)$ satisfying $Y \cap \{x \in R^n : \langle v, x\rangle \geq 1\} \neq \emptyset$. For every $v \in \Gamma_k \setminus V((S_{k-1})^\circ)$, let x^v be an optimal solution of problem $(SP(v))$. Choose $v^k \in \Gamma_k$ satisfying $f(x^{v^k}) = \min\{f(x^v) : v \in \Gamma_k$. Let $x(k) = x^{v^k}$.

Step 2.

a. If $p(x(k)) \geq 0$, then stop; $x(k)$ solves problem (MP) and the optimal value of problem $(SP(v^k))$ is the optimal value of problem (MP).

b. Otherwise, solve the following convex minimization problem:

$$\begin{cases} \text{minimize} & \phi(x\,;\,v^k) = \max\{p(x),\ h(x,\ v^k)\} \\ \text{subject to} & x \in R^n \end{cases} \tag{2}$$

where $h(x,\ v^k) = -\langle v^k,\ x\rangle + 1$. Let z^k denote an optimal solution of problem (2). It will be proved later in Theorems 3.1 and 3.2 that problem (2) has an optimal solution and that $z^k \in X$, respectively. Set $V_{k+1} = V_k \cup \{z^k\}$. Let $S_{k+1} =$co V_{k+1}. Compute the vertex set $V((S_{k+1})^\circ)$. Set $k \leftarrow k+1$ and return to Step 1.

Note that S_k, $k=1,2,\ldots$, are polytopes. Since $0 \in \mathrm{int}(\mathrm{co}\ V_1) = \mathrm{int}\ S_1$, S_k, $k=1,2,\ldots$, satisfy that $0 \in \mathrm{int}\ S_k$. It follows from the following theorems that at every iteration of the algorithm, problem (2) has an optimal solution and S_k is contained in X.

Theorem 3.1 [9] *For any $v \in R^n$, the function $\phi(x\,;\,v)$ attains its minimum over R^n.*

Theorem 3.2 [9] *At iteration k of Algorithm IA, assume that $S_k \subset X$. Then*

(i) $v \notin \mathrm{int}\ X^\circ$ *for any* $v \in V((S_k)^\circ)$.

(ii) $\phi(z^k\,;\,v^k) \leq 0$,

(iii) $z^k \in X$.

From Theorem 3.2 and the definition of S_1, we have

- $S_1 \subset S_2 \subset \cdots \subset S_k \subset \cdots \subset X$

- $(S_1)^\circ \supset (S_2)^\circ \supset \cdots \supset (S_k)^\circ \supset \cdots \supset X^0$.

Hence, for every iteration k of the algorithm, the following problems (P_k) and (D_k) are relaxed problems of (MP) and (DP), respectively.

$$(P_k) \begin{cases} \text{minimize} & g(x) \\ \text{subject to} & x \in R^n \setminus \text{int } S_k, \end{cases}$$

$$(D_k) \begin{cases} \text{maximize} & g^H(u) \\ \text{subject to} & u \in (S_k)^\circ. \end{cases}$$

From the discussion in Subsection 3.1, $x(k)$ and v^k obtained in Step 1 of the algorithm solve problems (P_k) and (D_k), respectively. Moreover, we note that $\max(D_{k-1}) \geq \max(D_k)$ for any $k \geq 2$, that is,

$$g^H(v^1) \geq g^H(v^2) \geq \cdots \geq g^H(v^k) \geq \cdots \geq \max(DP), \tag{3}$$

and that $\min(P_{k-1}) \leq \min(P_k)$ for any $k \geq 2$, that is,

$$g(x(1)) \leq g(x(2)) \leq \cdots \leq g(x(k)) \leq \cdots \leq \min(MP). \tag{4}$$

Since $g(x) = +\infty$ for any $x \notin Y$, $x(k)$ belongs to Y. It follows from the following theorem that $x(k)$ solves problem (MP) if $p(x(k)) \geq 0$.

Theorem 3.3 [9] *At iteration k of the algorithm, $x(k)$ solves problem (MP) if $p(x(k)) \geq 0$.*

For any k, the following assertions are valid.

- $V(S_k) \subset V_k$.
- $(S_k)^\circ = \{u \in R^n : \langle u, z \rangle \leq 1 \; \forall z \in V_k\}$.
- $(S_{k+1})^\circ = (S_k)^\circ \cap \{u \in R^n : \langle u, z^k \rangle \leq 1\}$.

Moreover, the following lemma holds.

Lemma 3.2 [9] *At iteration k of Algorithm IA, if $p(x(k)) < 0$, then $\langle v^k, z^k \rangle > 1$.*

From Lemma 3.2, $S_{k+1} = \text{co}(S_k \cup \{z^k\}) \neq S_k$ because $S_k \subset \{x \in R^n : \langle v^k, x \rangle \leq 1\}$ and $\langle v^k, z^k \rangle > 1$. Moreover, since $V(S_{k+1}) \subset V(S_k) \cup \{z^k\}$, we have

$$(S_{k+1})^\circ = (S_k)^\circ \cap \{u \in R^n : \langle u, z^k \rangle \leq 1\} \neq (S_k)^\circ \tag{5}$$

Remark 3.1 *At iteration k of Algorithm IA, for any $v \in V((S_{k+1})^\circ) \setminus V((S_k)^\circ)$, $\langle v, z^k \rangle = 1$.*

It follows from the following theorems that accumulation points of the sequences $\{x(k)\}$ and $\{v^k\}$ are optimal solutions of problem (MP) and (DP), respectively.

Theorem 3.4 [9] *Assume that $\{v^k\}$ is an infinite sequence such that for all k, v^k is an optimal solution of (D_k) at iteration k of Algorithm IA and that $\bar{v}$ is an accumulation point of $\{v^k\}$. Then $\bar{v}$ solves problem (DP). Furthermore, $\lim_{k\to\infty} g^H(v^k) = \max(DP)$.*

Theorem 3.5 [9] *Assume that* $\{x(k)\}$ *is an infinite sequence such that for all* k, $x(k)$ *is an optimal solution of problem* (P_k) *at iteration* k *of Algorithm IA and that* $\bar{x}$ *is an accumulation point of* $\{x(k)\}$. *Then* $\bar{x}$ *belongs to* $R^n \setminus \mathrm{int}\ X$ *and solves problem* (MP). *Furthermore,* $\lim_{k\to\infty} g(x(k)) = \min(MP)$.

4 An Inner Approximation Method Incorporating with a Penalty Function Method

4.1 Underestimation of the Optimal Value of Relaxed Problems by Using Penalty Functions

In order to obtain an optimal solution of problem (P_k), problem $(SP(v))$ has been solved for each $v \in \Gamma_k \setminus V((S_{k-1})^\circ)$ at every iteration of Algorithm IA discussed in Section 3. In Subsection 3.1, we remarked that problem $(SP(v))$ is a convex minimization problem with convex constraints. In this section, we explain another inner approximation algorithm incorporating with a penalty function method proposed by Yamada, Tanino and Inuiguchi [9]. By using penalty functions, problem $(SP(v))$ can be transformed into an unconstrained convex minimization problem. That is, without solving problem $(SP(v))$ at every iteration, the algorithm guarantees the global convergence to an optimal solution of problem (MP). Furthermore, the problem is solvable for every $v \in V((S_k)^\circ)$. Hence, by incorporating with a penalty function method, the inner approximation algorithm does not need to generate Γ_k at every iteration.

Let $S \subset X$ be a polytope satisfying $0 \in \mathrm{int}\ S$. For any $v \in V(S)$, we consider the following problem:

$$(SP1(v,\mu)) \quad \begin{cases} \text{minimize} & F_{v,\mu}(x) = f(x) + \mu\,\theta_v(x), \\ \text{subject to} & x \in R^n, \end{cases}$$

where $\theta_v(x) = \Sigma_{j=1}^{r}[\max\{0, r_j(x)\}]^s + [\{\max\{0, h(x, v)\}]^s$, $s \geq 1$ and $\mu > 0$. We know that the objective function $F_{v,\mu}$ of problem $(SP1(v,\mu))$ is convex (Bazaraa, Sherali and Shetty [1], Chapter 9). It follows from the following lemma that problem $(SP1(v,\mu))$ is solvable for every $v \in V(S^\circ)$.

Lemma 4.1 [9] *For every* $v \in R^n$ *and* $\mu > 0$, *the function* $F_{v,\mu}$ *attains its minimum over* R^n. Denote by $\min(SP1(v,\mu))$ the optimal value of problem $(SP1(v,\mu))$. From the definition of g, $\min(SP1(v,\mu)) < -g^H(v) = +\infty$ if $v \notin \Gamma$. In case $v \in \Gamma$, since $F_{v,\mu}(x) = f(x)$ for any $x \in Y \cap \{x \in R^n : \langle v,x \rangle \geq 1\}$,

$$\begin{aligned} \min\ (SP1(v,\mu)) &= \min\{F_{v,\mu}(x) : x \in R^n\} \\ &\leq \min\{F_{v,\mu}(x) : \langle v, x \rangle \geq 1,\ x \in Y\} \\ &= \min\{f(x) : \langle v, x \rangle \geq 1, x \in Y\} \\ &= \min(SP(v)) \end{aligned} \tag{6}$$

$$= -g^H(v).$$

Hence, we have the following relations between problem $(SP1(v, \mu))$ and relaxed problems (P) and (D) described in Subsection 3.1:

$$\begin{aligned}\min(P) &= \min\{\min(SP(v)) : v \in \Gamma\}\\ &\geq \min\{\min(SP1(v,\mu)) : v \in \Gamma\} \qquad (7)\\ &\geq \min\{\min(SP1(v,\mu)) : v \in V(S^\circ)\},\end{aligned}$$

and

$$\begin{aligned}\max(D) &= \max\{g^H(v) : v \in V(S^\circ)\}\\ &\leq \max\{-\min(SP1(v,\mu)) : v \in V(S^\circ)\}.\end{aligned} \qquad (8)$$

4.2 An Inner Approximation Algorithm Using Penalty Functions

An inner approximation algorithm for problem (MP) incorporating with an exterior penalty method is as follows:

Algorithm IA-P

Initialization. Choose a penalty parameter $\mu_1 > 0$, a scalar $B > 1$ and $s \geq 1$. Generate a polytope V_1 such that $V_1 \subset X$ and that $0 \in \mathrm{int}(\mathrm{co}\ V_1)$. Let $S_1 = \mathrm{co}\ V_1$. Compute the vertex set $V((S_1))$. Set $k \leftarrow 1$ and go to Step1.

Step 1. For every $v \in V((S_k)^\circ)$, let x^v be an optimal solution of problem $(SP1(v, \mu_k))$. Choose $v^k \in \arg\min\{F_{v,\mu_k}(x^v) : v \in V((S_k)^\circ)\}$. Let $x(k) = x^{v^k}$.

Step 2.

a. If $p(x(k)) \geq 0$ and $r(x(k)) \leq 0$, then stop; $x(k)$ is an optimal solution of problem (MP).

b. Otherwise, for v^k, solve problem (2). Let z^k and ω_k denote an optimal solution and the optimal value of problem (2), respectively. Let

$$V_{k+1} = \begin{cases} V_k \cup \{z^k\} & \text{if } \omega_k < 0,\\ V_k & \text{if } \omega_k = 0,\end{cases}$$

and let

$$\mu_{k+1} = \begin{cases} B\mu_k & \text{if } \phi_{v_k}(x(k)) > 0,\\ \mu_k & \text{if } \phi_{v_k}(x(k)) = 0.\end{cases}$$

Let $S_{k+1} = \mathrm{co}\ V_{k+1}$. Compute the vertex set $V((S_{k+1})^\circ)$. Replace k by $k+1$, and return to Step 1.

From the discussion of Subsection 4.1, at every iteration k of the algorithm, we have

$$f(x(k)) \leq F_{v^k,\mu_k}(x(k)) \leq \min(P_k) \leq \min(MP). \qquad (9)$$

Theorem 4.1 [9] *At iteration k of Algorithm IA-P, if* $p(x(k)) \geq 0$ *and* $r(x(k)) \leq 0$, *then* $x(k)$ *solves problem* (MP).

It follows from the following theorems that accumulation points of the sequences $\{x(k)\}$ and $\{v^k\}$ are optimal solutions of problem (MP) and (DP), respectively.

Theorem 4.2 *Let* $\{x(k)\}$ *be an infinite sequence generated by Algorithm IA-P. Then, every accumulation point* $\bar{x}$ *of* $\{x(k)\}$ *solves problem* (MP).

Theorem 4.3 *Let* $\{v^k\}$ *be an infinite sequence generated by Algorithm IA-P. Then, every accumulation point* $\bar{v}$ *of* $\{v^k\}$ *solves problem* (DP).

5 An Exact Penalty Parameter of an Inner Approximation Method Using Penalty Functions

In this section, we assume $s=1$ at Initialization of Algorithm IA-P. Then, we shall show that there exist an exact penalty parameter $\Lambda >$ and $\bar{k}$ such that for each $\mu \geq \Lambda$ and $k \geq \bar{k}$, every optimal solution of problem $(SP1(v^k, \mu))$ solves problem $(SP(v^k))$.

Let Ω_M and Ω_D be the optimal solution sets of problems (MP) and (DP), respectively.

Lemma 5.1 *Assume that* $\nabla f(x') \neq 0$ *for some* $x' \in \Omega_M$. *Then, for any* $u \in \Omega_D$, $\{y \in R^n : \langle u, y \rangle \geq 1\} \cap \mathrm{int}\ Y \neq \emptyset$.

Proof. Since $\nabla f(x') \neq 0$ for some $x' \in \Omega_M$, $\nabla f(x) \neq 0$ for any $x \in \Omega_M$. By assumption (A4), $\Omega_M \subset \mathrm{bd}\ X$. From the definition of g and g^H, $g^H(0) = -\sup\{g(y) : y \in R^n\} = -\infty$, that is, $0 \notin \Omega_D$. Hence, $g^H(u) = -\inf\{g(y) : \langle u, y \rangle \geq 1\} = -\inf\{f(y) : \langle u, y \rangle \geq 1,\ y \in Y\}$ for any $u \in \Omega_D$. Since $\max(DP) > -\infty$, $\{y \in R^n : \langle u, y \rangle \geq 1\} \cap \Omega_M \neq 0$ for any $u \in \Omega_D$, i.e., there exists $x(u) \in \Omega_M$ such that $\langle u, x(u) \rangle \geq 1$. Moreover, since $\Omega_D \subset X^0$ and $\Omega_M \subset \mathrm{bd}\ X \cap Y$, for any $u \in \Omega_D$, $X \subset \{y \in R^n : \langle u, y \rangle \leq 1\}$ and $\{y \in R^n : \langle u, y \rangle = 1\}$ is a supporting hyperplane of X at $x(u)$. Therefore, by assumption (A5), $\{y \in R^n : \langle u, y \rangle \geq 1\} \cap \mathrm{int}\ Y \neq 0$ for any $u \in \Omega_D$. □

For any $u \in (S_1)^0 \setminus \mathrm{int}\ Y^0$, let $\Omega_{(SP(u))}$ be the optimal solution set of problem $(SP(u))$.

Lemma 5.2 $\Omega_M = \bigcup_{u \in \Omega_D} \Omega_{(SP(u))}$.

Proof. We shall show that $\Omega_M \supset \bigcup_{u \in \Omega_D} \Omega_{(SP(u))}$. Since $\Omega_D \subset X^0$, $\bigcup_{u \in \Omega_D} \Omega_{(SP(u))} \subset R^n \setminus \mathrm{int}\ X$. From the definition of g^H, $g(x) = +\infty$ for any $x \notin Y$, so that $0 \in R^n$ is not optimal to problem (DP). Hence, from the duality of problems (MP) and (DP), we get that for any $u \in \Omega_D$,

$$\min(MP) = -\max(DP)$$

$$
\begin{aligned}
&= -g^H(u) \\
&= \inf\{g(x) : \langle u, x\rangle \geq 1\} \\
&= \inf\{f(x) : \langle u, x\rangle \geq 1,\ x \in Y\} \\
&= \min(SP(u)).
\end{aligned}
$$

Therefore, $\Omega_M \supset \cup_{u\in\Omega_D}\Omega_{(SP(u))}$.

Conversely, since $\Omega_M \subset R^n \setminus \mathrm{int}\ X$, for any $x' \in \Omega_M$, there exists $u' \in X$ such that $\langle u', x'\rangle \geq 1$. Then, we have $g^H(u') \leq \max(DP)$ and

$$
\begin{aligned}
g^H(u') &= -\inf\{g(x) : \langle u', x\rangle \leq 1\} \\
&\geq -g(x') \\
&= -\min(MP) \\
&= \max(DP).
\end{aligned}
$$

Hence, we get that $u' \in \Omega_D$ and $\inf\{g(x) : \langle u', x\rangle \geq 1\} = g(x')$. Since $\Omega_M \subset Y$ and $\inf\{g(x) : \langle u', x\rangle \geq 1\} = \inf\{f(x) : \langle u', x\rangle \geq 1,\ x \in Y\}$, x' is an optimal solution of problem $(SP(u'))$. Therefore, $\Omega_M \subset_{u\in\Omega_D}\Omega_{(SP(u))}$. □

Lemma 5.3 *Assume that* $\nabla f(x') \neq 0$ *for some* $x' \in \Omega_M$. *Then, for any* $u \in \Omega_D$, $\Omega_{(SP(u))} \subset \{x \in R^n : \langle u, x\rangle = 1\}$.

Proof. Since $\nabla f(x') \neq 0$ for some $x' \in \Omega_M$, $\nabla f(x) \neq 0$ for any $x \in \Omega_M$. From Lemma 5.2, $\nabla f(x) \neq 0$ for any $u \in \Omega_D$ and $x \in \Omega_{(SP(u))}$. Assume that $\dot{x} \notin \{x \in R^n : \langle \dot{u}, x\rangle = 1\}$ for some $\dot{u} \in \Omega_D$ and $\dot{x} \in \Omega_{(SP\dot{u}))}$. Since $0, \dot{x} \in Y$, $\langle \dot{u}, 0\rangle = 0 < 1$ and $\langle \dot{u}, \dot{x}\rangle > 1$, we get that there exists $\ddot{x} \in \{x \in Y : \langle \dot{u}, x\rangle = 1\} \cap]0, \dot{x}[$. Since $f(\dot{x}) > f(0)$, from assumption (A4) and the convexity of f, $f(y) < f(\dot{x})$ for any $y \in]0. \dot{x}[$. This implies that $f(\ddot{x}) < f(\dot{x})$. Since $\ddot{x} \in \{x \in Y : \langle \dot{u}, x\rangle \geq 1\}$, this contradicts the optimality of $\dot{x}$ to problem $(SP(\dot{u}))$. Consequently, $\Omega_{(SP(u))} \subset \{x \in R^n : \langle u, x\rangle = 1\}$ for any $u \in \Omega_D$. □

For any $u \in (S_1)^0 \setminus \mathrm{int}\ Y$, let $Y(u) = \{x \in Y : -\langle u, x\rangle + 1 \leq 0\}$. Then, $Y(u)$ is the feasible set of problem $(SP(u))$. Moreover, let $r(u, x) = \max\{r_j(u, x) : j=1,\ldots, t_Y+1\}$ where $r_j(u, x) = r_j(x)$, $j=1,\ldots,t_Y$ and $r_{t_{Y+1}}(u, x) = -\langle u, x\rangle + 1$, and let $\partial_x r(u, x) = \mathrm{co}(\nabla r_1(x),\ldots, r_{t_Y}(x), -u)$. Note that $Y(u) = \{x \in R^n : r_u(x) \leq 0\}$.

Lemma 5.4 *For any* $u \in \Omega_D$ *and* $x \in \mathrm{bd}\ Y(u)$, $0 \notin \partial_x r(u, x)$.

Proof. From Lemma 5.1, for any $u \in \Omega_D$, $\{y \in R^n : \langle u, y\rangle \geq 1\} \cap \mathrm{int}\ Y \neq \emptyset$. Hence, by assumption (A3), for any $u \in \Omega_D$, there exists $y(u) \in Y(u)$ such that $r(u, y(u)) < 0$. Since $r(u, \cdot)$ is a convex function for each u, for any $y \in \mathrm{bd}\ Y$ and $a \in \partial_x r(u, y)$, we have

$$0 > r(u, y(u))\ t \geq r(u, y) + \langle a, y(u) - y\rangle = \langle a, y(u) - y\rangle.$$

The proof is complete. □

Lemma 5.5 *Let* $\Upsilon\subset R^n\setminus\mathrm{int}\,X$ *satisfy that* $(\mathrm{int}\,Y)\cap\{x\in R^n:\langle u,x\rangle\geq 1\}\neq\emptyset$ *for all* $u\in\Upsilon$. *Then,* $\Omega_{(SP(u))}$ *is upper semicontinuous over* Υ.

Proof. Since f is continuous, Y is a closed set and $Y(u)=\mathrm{cl}\{x\in Y:\langle u,x\rangle>1\}$ for any $u\in\Upsilon$, the point-to-set map $\Omega_{(SP(u))}$ is closed on Υ. Moreover, from the definition of Υ, for any $u\in\Upsilon$, there exists $x_u\in\mathrm{int}\,Y$ such that $\langle u,x_u\rangle>1$. Then, there exists a neighborhood $N(u)$ of u such that $\langle u',x_u\rangle>1$ for any $u'\in N(u)\cap\Upsilon$. Therefore, $\min(SP(u'))\leq f(x_u)$ for any $u'\in N(u)\cap\Upsilon$, so that $\bigcup_{u'\in N(u)\cap\Upsilon}\Omega_{(SP(u'))}\subset\{x\in R^n:f(x)\leq f(x_u)\}$. From assumption (A2), $\{x\in R^n:f(x)\leq f(x_u)\}$ is compact. That is, $\Omega_{(SP(u))}$ is uniformly compact and upper semicontinuous on Υ. □

Lemma 5.6 *The point-to-set map* $\bigcup_{x\in\Omega_{(SP(u))}}\partial_x r(u,x)$ *is upper semicontinuous over* Υ.

Proof. We shall show that for any $\bar{u}\in\Upsilon$ and $\varepsilon>0$, there exists $\delta>0$ such that

$$\bigcup_{x\in\Omega_{(SP(u))}}\partial_x r(u,x)\subset\Big(\bigcup_{x\in\Omega_{(SP(\bar{u}))}}\partial_x r(\bar{u},x)\Big)+\varepsilon U,\qquad\forall u\in B(\bar{u},\delta),\tag{10}$$

where U is the Euclidean unit ball in R^n. Since $\nabla r_j(x)$ $(j=1,\ldots,t_Y)$ are continuous on R^n, for any $\bar{x}\in\Omega_{(SP(\bar{u}))}$, there exists $\alpha_1(\bar{x})>0$ such that

$$\|\nabla r_j(x)-\nabla r_j(\bar{x})\|<\varepsilon,\ j=1,\ldots,t_Y,\ \ \forall x\in B(\bar{x},\ \alpha_1(\bar{x})).\tag{11}$$

Let $J_{\bar{u}}(\bar{x})=\{j:r_j(\bar{u},\bar{x})=r(\bar{u},\bar{x}),\ j=1,\ldots,t_Y+1\}$, and $\beta_{\bar{x}}:=\max\{r_j(\bar{u},\bar{x}):j\in\{1,\ldots,t_Y+1\}\setminus J_{\bar{u}}(\bar{x})\}<r(\bar{u},\bar{x})$. Then, since $r(\bar{u},x)$ and $r_j(\bar{u},x)$, $j=1,..,t_Y$, are continuous, there exists $\alpha_2(\bar{x})>0$ such that

$$r(\bar{u},x)>\frac{1}{2}(r(\bar{u},\bar{x})+\beta_{\bar{x}})\ \ and\ \ \max_{j\in\{1,\ldots,t_Y\}\setminus J_{\bar{u}}(\bar{x})}r_j(\bar{u},x)<\frac{1}{2}r(\bar{u},\bar{x})+\beta_{\bar{x}}),\quad\forall x\in B(\bar{x},\alpha_2(\bar{x})).\tag{12}$$

Moreover, since $\langle\cdot,\cdot\rangle$ is continuous on $R^n\times R^n$, there exist $\alpha_3(\bar{x})>0$ and $\delta_1>0$ such that

$$\begin{aligned}r_{t_Y+1}(u,x)&=-\langle u,x\rangle+1>\frac{1}{2}(r(\bar{u},\bar{x})+\beta_{\bar{x}})\quad\forall x\in B(\bar{x},\alpha_3(\bar{x})),\ \ u\in B(\bar{u},\delta_1)\cap\Upsilon\\&\qquad\text{if }t_Y+1\in J_{\bar{u}}(\bar{x}),\\r_{t_Y+1}(u,x)&=-\langle u,x\rangle+1<\frac{1}{2}r(\bar{u},\bar{x})+\beta_{\bar{x}})\quad\forall x\in B(\bar{x},\alpha_3(\bar{x})),\ u\in B(\bar{u},\delta_1)\cap\Upsilon\\&\qquad\text{otherwise.}\end{aligned}\tag{13}$$

Let $\alpha(\bar{x})=\min\{\alpha_1(\bar{x}),\ \alpha_2(\bar{x}),\ \alpha_3(\bar{x})\}$. From assumption (A2), $\Omega_{(SP(\bar{u}))}$ is compact, so that there exists $\bar{x}^1,\ldots,\bar{x}^L\in\Omega_{(SP(\bar{u}))}$ such that

$$\Omega_{(SP(\bar{u}))}\subset\bigcup_{l=1}^{L}B(\bar{x}^l,\alpha(\bar{x}^l))\subset\bigcup_{x\in\Omega_{(SP(\bar{u}))}}B(\bar{x},\alpha(\bar{x})).\tag{14}$$

By (14), there exists $\alpha>0$ such that

$$\Omega_{(SP(\overline{u}))} + \alpha \bigcup_{l=1}^{L} \subset B(\overline{x}^1, \alpha(\overline{x}^1)) \subset \bigcup_{x \in \Omega_{(SP(\overline{u}))}} B(\overline{x}^1, \alpha(\overline{x})). \tag{15}$$

From Lemma 5.5, $\Omega_{(SP(u))}$ is upper semicontinuous on Υ. Hence, there exists $\delta_2 > 0$ such that

$$\Omega_{(SP(u))} \subset \Omega_{(SP(\overline{u}))} + \alpha U, \qquad \forall u \in B(\overline{u}, \delta_2) \cap \Upsilon. \tag{16}$$

Let $\delta = \min\{\delta_1, \delta_2, \varepsilon\}$. Then, $\| u - \overline{u} \| < \varepsilon$ for any $u \in B(\overline{u}, \delta) \cap \Upsilon$. By (11), (12), (13), (14), (15) and (16), for any $u \in B(\overline{u}, \delta) \cap \Upsilon$ and $x \in \Omega_{(SP(u))}$, there exists $\hat{x} \in \Omega_{(SP(\overline{u}))}$ such that

$$\| \nabla r_j(u,x) - \nabla r_j(\overline{u,x}) \| < \varepsilon, \; j=1,\dots, t_Y+1. \tag{17}$$

and that for all $j \in \{1,\dots, t_Y+1\}$,

$$\begin{aligned} &r_j(u,x) > \frac{1}{2}(r(\overline{u},\hat{x}) + \beta_{\hat{x}}) \;\text{ if } j \in J_{\overline{u}}(\hat{x}), \\ &r_j(u,x) < \frac{1}{2}(r(\overline{u},\hat{x}) + \beta_{\hat{x}}) \quad \text{otherwise.} \end{aligned} \tag{18}$$

By (18), $J_u(x) \subset J_{\overline{u}}(\hat{x})$. Consequently, $\partial_x r(u,x) \subset \partial x r(\overline{u},\hat{x}) + \varepsilon U$.

Lemma 5.7 *Assume that* $\nabla f(x) \neq 0$ *for some* $x' \in \Omega_M$. *Then, the following assertion holds:*

$$\inf \left\{ \| w \| : w \in \bigcup_{u \in \Omega_D} \Big(\bigcup_{x \in \Omega_{(SP(u))}} \partial_x r(u,x) \Big) \right\} > 0.$$

Proof. It follows from assumption (A2) that $\Omega_{(SP(\overline{u}))}$ is compact for each $\overline{u} \in \Omega_D$. Hence, $\bigcup_{x \in \Omega_{(SP(\overline{u}))}} (\partial_x r(\overline{u},x))$ is compact (Rockafellar[7], Theorem 24.7). Hence, from Lemmas 5.3 and 5.4,

$$\inf \left\{ \| w \| : w \in \bigcup_{x \in \Omega_{(SP(\overline{u}))}} \partial_x r(\overline{u},x) \right\} > 0. \tag{19}$$

For any $\overline{u} \in \Omega_D$, let $\varepsilon(\overline{u}) = (1/2)\inf\{ \| w \| : w \in \bigcup_{x \in \Omega_{(SP(\overline{u}))}} \partial_x r(\overline{u},x)\}$. From Lemma 5.6, for any $\overline{u} \in \Omega_D$, there exists $\delta(\overline{u}) > 0$ such that

$$\bigcup_{x \in \Omega_{(SP(u))}} (\partial_x r(u,x)) \subset \bigcup_{x \in \Omega_{(SP(\overline{u}))}} (\partial_x r(\overline{u},x)) + \varepsilon(\overline{u}) U, \quad \forall u \in B(\overline{u}, \delta(\overline{u})) \cap \Omega_D.$$

From the compactness of Ω_D, there exists $\overline{u}^1,\dots,\overline{u}^L \in \Omega_D$ such that

$$\Omega_D \bigcup_{l=1}^{L} B(\overline{u}^l, \delta(\overline{u}^l)). \tag{21}$$

By (20) and (21), we have

$$\bigcup_{u \in \Omega_D} \Big(\bigcup_{x \in \Omega_{(SP(u))}} \partial_x r(u,x) \Big) \subset \bigcup_{\ell=1}^{L} \Big(\bigcup_{x \in \Omega_{(SP(\overline{u}))}} (\partial_x r(\overline{u}^l,x)) + \varepsilon(\overline{u}^l) U \Big). \tag{22}$$

Let $\varepsilon = \min\{\varepsilon(\overline{u}) : l=1,\dots, L\} > 0$. Then, we have

$$\varepsilon U \cap \left(\bigcup_{\ell=1}^{L} \left(\bigcup_{x \in \Omega_{(SP(\bar{u}^\ell))}} (\partial_x r(\bar{u}^\ell, x)) + \varepsilon (\bar{u}^\ell) U \right) \right) = \emptyset.$$

The proof is complete. □

Theorem 5.1 *There exists* $\Lambda \geq 0$ *such that for any* $u \in \Omega_D$ *and* $x \in \Omega_{(SP(v))}$, *there exists* $\lambda(u,x)_j \geq 0$ $(j=1, \ldots, t_Y+1)$ *satisfying*

$$\max_{j=1,\ldots,t_Y+1} \lambda(u,x)_j \leq \Lambda, \tag{23}$$

$$\nabla f(x) + \sum_{j=1}^{t_Y} \lambda(u,x)_j \nabla r_j(x) - \lambda(u,x)_{t_Y+1} u = 0, \tag{24}$$

$$\lambda(u,x)_j r_j(y) = 0 \ (j=1,\ldots, t_Y) \text{ and } \lambda(u,x)_{t_Y+1}(-\langle u,x\rangle + 1) = 0. \tag{25}$$

Proof. Note that for each $u \in \Omega_D$, problem $(SP(u))$ is a convex programming problem. Hence, for any $u \in \Omega_D$ and $x \in \Omega_{(SP(u))}$, there exist $\lambda(u,x)_j \geq 0$ $(j=1,\ldots, t_Y+1)$ satisfying conditions (24) and (25). In case $\nabla f(x')=0$ for some $u' \in \Omega_D$ and $x' \in \Omega_{(SP(u'))}$, $\nabla f(x)=0$ for any $u \in \Omega_D$ and $x \in \Omega_{(SP(u))}$. Then, $\lambda(u,x)_j = 0$ $(j=1,\ldots, t_Y+1)$ satisfy conditions (24) and (25). Therefore, $\Lambda=0$ satisfies condition (23).

In the other case, for any $u \in \Omega_D$ and $x \in \Omega_{(SP(u))}$, since x is an optimal solution of problem $(SP(u))$ and problem $(SP(u))$ is a convex programming problem, $-\nabla f(x) \in N_{Y(u)}(x)$. By assumptions (A3) and (A5), we note that for any $u \in \Omega_D$ and $x \in \Omega_{(SP(u))}$, $N_{Y(u)}(x) = \{y \in R^n : y = \lambda w,\ \lambda \geq 0,\ w \in \partial r_u(x)\}$. From Lemma 5.2, we have $\bigcup_{u \in \Omega_D} \Omega_{(SP(u))} = \Omega_M$, that is, $\bigcup_{u \in \Omega_D} \Omega_{(SP(u))}$ is compact. Let $\alpha^f_{\max} = \{ \| \nabla f(x) \| : x \in \Omega_M \} > 0$. Since $\nabla f(x) \neq 0$ for any $x \in \Omega_{(SP(u))}$ and $u \in \Omega_D$, $\Omega_{(SP(u))} \subset \operatorname{bd} Y(u)$. Hence, by Lemma 5.7,

$$\inf \left\{ \| w \| : w \in \bigcup_{u \in \Omega_D} \left(\bigcup_{x \in \Omega_{(SP(u))}} \partial r_u(x) \right) \right\} > 0.$$

Let $\alpha_{\inf} = \inf \| w \| : w \in \bigcup_{u \in \Omega_D} (\bigcup_{x \in \Omega_{(SP(u))}} \partial r_u(x))\} > 0$ and $\alpha = \alpha^f_{\max} / \alpha_{\inf} > 0$. Then, we note that for any $u \in \Omega_D$ and $x \in \Omega_{(SP(u))}$,

$$-\nabla f(x) \in \{y \in R^n : y = \lambda w,\ 0 \leq \lambda \leq \alpha,\ w \in \partial r_u(x)\},$$

and

$$\begin{aligned}
&\{y \in R^n : y = \lambda w,\ 0 \leq \lambda \leq \alpha,\ w \in \partial r_u(x)\} \\
&= \{y \in R^n : y = \lambda \sum_{j=1}^{t_Y} \mu_j \nabla r_j(x) - \mu_{t_Y+1} u,\ 0 \leq \lambda \leq \alpha, \\
&\qquad \sum_{j=1}^{t_Y+1} \mu_j = 1,\ \mu_j \geq 0,\ j=1,\ldots, t_Y+1\} \\
&\subset \{y \in R^n : y = \sum_{j=1}^{t_Y} \lambda_j \nabla r_j(x) - \lambda_{t_Y+1} u,\ 0 \leq \lambda_j \leq \alpha,\ j=1,\ldots, t_Y+1\}.
\end{aligned}$$

This implies that $\Lambda=\alpha$ satisfies condition (23). □

Lemma 5.8 *Let $\{v^k\}$ be generated by Algorithm IA-P. Assume that $\nabla f(x')\neq 0$ for some $x' \in \Omega_M$. Then, there exists $\bar{k}$ such that $Y\cap\{x\in R^n : \langle v^k,x\rangle\geq 1\}\neq\emptyset$ for all $k\geq\bar{k}$.*

Proof. Since $\{v^k\}\subset(S_1)^0$ and (S_1) is compact, without loss of generality, we can assume that $\{v^k\}$ converges to $\bar{v}$. By Theorem 4.3, $\bar{v}\in\Omega_D$. From the assumption of this lemma and Lemma 5.1, for any $u\in\Omega_D$, $\{y\in R^n : \langle u,y\rangle\geq 1\}\cap$ int $Y\neq\emptyset$. Hence, there exists $\bar{x}\in\{y\in R^n : \langle\bar{v},y\rangle>1\}\cap$ int Y. Then, we have

$$\lim_{k\to\infty}\langle v^k,\bar{x}\rangle=\langle\bar{v},\hat{x}\rangle>1.$$

That is, there exists $\bar{k}$ such that $\langle v^k,\hat{x}\rangle>1$ for all $k>\bar{k}$. The proof is complete. □

Theorem 5.2 *Assume that $s=1$ at Initialization of Algorithm IA-P. Then, there exists $\bar{k}$ such that $\theta_{v^k}(x(k))=0$ for all $k\geq\bar{k}$. Furthermore, for all $k\geq\bar{k}$, an optimal solution $x(k)$ of problem $(SP1(v^k,\mu_k))$ solves problem $(SP(v^k))$.*

Proof. In case $\nabla f(x')=0$ for some $x'\in\Omega_D$, from assumption (A4) and (9), it follows that for all k, $f(0)=f(x(k))=F_{v^k,\mu_k}(x(k))=\min(SP1(v^k,\mu_k))=\min(MP)$. Since $\mu_k\theta_{v^k}(x)>0$ for any $x\notin Y(v^k)$, $F_{v^k,\mu_k}(x)=f(x)+\mu_k\theta_{v^k}(x)\geq f(0)+\mu_k\theta_{v^k}(x)>f(0)$ for any $x\notin Y(v^k)$. This implies that $Y(v^k)\neq\emptyset$ and that problem $(SP1(v^k,\mu_k))$ is equivalent to the following problem:

$$\begin{cases}\text{minimize} & F_{v^k\mu_k}(x),\\ \text{subject to} & x\in Y(v^k).\end{cases}\tag{26}$$

Moreover, since $F_{v^k,\mu_k}(x)=f(x)$ for any $x\in Y(v^k)$, problem (26) is equivalent to problem $(SP(v^k))$, that is, problem $(SP1(v^k,\mu_k))$ is equivalent to problem $(SP(v^k))$ for all $k\geq 1$.

In the other case, since $\{v^k\}\subset(S_1)$ and (S_1) is compact, without loss of generality, we can assume that $\{v^k\}$ converges to $\bar{v}$. By Theorem 4.3, $\bar{v}\in\Omega_D$. From, Lemma 5.1, $\{x\in R^n : \langle\bar{v},x\rangle\geq 1\}\cap$ int $Y\neq 0$. Let $x'\in\{x\in R^n : \langle\bar{v},x\rangle>1\}\cap$ int Y. Then, there exists $\delta_1>0$ such that $\langle u,x'\rangle>1$ for any $u\in B(\bar{v},\delta_1)$, so that $\{x\in R^n : \langle u,x\rangle>1\}\cap$ int $Y\neq 0$ for any $u\in B(\bar{v},\delta_1)$. Let $\Psi(u)=\max\{\langle u,x\rangle : f(x)=f(0),\ x\in Y\}$. Then, since $\nabla f(x)=0$ for any $x\in\{x\in R^n : f(x)=f(0),\ x\in Y\}$ and $\nabla f(x)\neq 0$ for any $x\in\Omega_{(SP(\bar{v}))}$, from the convexity of f, $\Psi(\bar{v})<1$. From assumption (A2), $\{x\in R^n : f(x)=f(0),\ x\in Y\}$ is compact. Therefore, Ψ is continuous. Hence, there exists $\delta_2>0$ such that $\Psi(u)<1$ for any $u\in B(\bar{v},\delta_2)$, so that $\{x\in R^n : \langle u,x\rangle\geq 1\}\cap\{x\in Y : f(x)=f(0)\}=\emptyset$ for any $u\in B(\bar{v},\delta_2)$. Let $\delta_3=\min\{\delta_1,\delta_2\}$. Then, $\Omega_{(SP(u))}\subset$ bd $Y(u)$, $0\notin\partial_x r(u,x)$ and $\nabla f(x)\neq 0$ for all $u\in B(\bar{v},\delta_3)$ and x

$\in \Omega_{(SP(u))}$.

Now, we shall show that for any $\varepsilon > 1$, there exists $\bar{k}(\varepsilon) \geq 0$ such that for all $k \geq k(\varepsilon)$ and $x \in \Omega_{(SP(v^k))}$, there exists $\lambda(v^k, x)_j \geq 0$ $(j = 1, \ldots, t_Y + 1)$ satisfying conditions (24), (25) in Theorem 5.1 and

$$\max_{j=1,\ldots,t_Y+1} \lambda(v^k, x)_j \leq \varepsilon \Lambda,$$

where Λ satisfies condition (23) in Theorem 5.1. Let

$$\alpha(u)^f_{\max} = \max\{\|\nabla f(x)\| : x \in \Omega_{(SP(u))}\},$$

$$\alpha(u)_{\inf} = \inf \left\{ \|w\| : w \in \bigcup_{x \in \Omega_{(SP(u))}} \partial_x r(u, x) \right\}.$$

Then, we get that $\alpha(u)^f_{\max} > 0$ and $\alpha(u)_{\inf} > 0$ for any $u \in B(\bar{v}, \delta_3)$. From Lemma 5.6, since $\bigcup_{x \in \Omega_{(SP(u))}} \partial_x r(u, x)$ is upper semicontinuous on $B(\bar{v}, \delta_3)$, there exists $\delta_1(\varepsilon) > 0$ $(\delta_1(\varepsilon) \leq \delta_3)$ such that

$$\bigcup_{x \in \Omega_{(SP(u))}} \partial_x \in \Omega_{(SP(u))} \subset \Big(\bigcup_{x \in \mathrm{bd}\, Y(\bar{v})} \partial_x r(u, x) \Big) + \frac{\varepsilon - 1}{\varepsilon + 1} \alpha(\bar{v})_{\inf} U, \quad \forall u \in B(\bar{v}, \delta_1(\varepsilon)). \tag{28}$$

Moreover, by Lemma 5.5, $\Omega_{(SP(u))}$ is upper semicontinuous over $B(\bar{v}, \delta_3)$, so that there exists $\delta_2(\varepsilon) > 0$ $(\delta_2(\varepsilon) \leq \delta_3)$ such that

$$\nabla f(x) \in \Big(\bigcup_{x \in \Omega_{(SP(\bar{v}))}} \nabla f(x) \Big) + \frac{\varepsilon - 1}{\varepsilon + 1} \alpha(\bar{v})^f_{\max} U, \quad \forall u \in B(\bar{v}, \delta_2(\varepsilon)) \text{ and } x \in \Omega_{(SP(u))}. \tag{29}$$

Let $\delta(\varepsilon) = \min\{\delta_1(\varepsilon), \delta_2(\varepsilon)\}$. From the proof of Theorem 5.1, we obtain

$$\frac{\alpha(\bar{v})^f_{\max}}{\alpha(\bar{v})_{\inf}} \leq \Lambda \tag{30}$$

By (28), (29) and (30) for any $u \in B(\bar{v}, \delta(\varepsilon))$,

$$\frac{\alpha(u)^f_{\max}}{\alpha(\bar{v})_{\inf}} < \frac{\alpha(u)^f_{\max} + \frac{\varepsilon-1}{\varepsilon+1} \alpha(\bar{v})^f_{\max}}{\alpha(\bar{v})_{\inf} - \frac{\varepsilon-1}{\varepsilon+1} \alpha(\bar{v})_{\inf}} = \frac{2\varepsilon\, \alpha(\bar{v})^f_{\max}}{2\alpha(\bar{v})_{\inf}} \leq \varepsilon \Lambda \tag{31}$$

Therefore, for any $\varepsilon > 0$, there exists $k(\varepsilon) > 0$ satisfying condition (27).

In order to obtain a contradiction, suppose that there is no $\bar{k}$ such that $\phi_{v^k}(x(k)) = 0$ for any $k \geq \bar{k}$. Without loss of generality, we can assume that $\phi_{v^k}(x(k)) > 0$ for all k. Then, $\lim_{k\to\infty} v^k = \bar{x}$ and $\lim_{k\to\infty} \mu_k = +\infty$. Hence, there exist $\varepsilon > 0$ and $k(\varepsilon) > 0$ such that $\varepsilon\Lambda \leq \mu_{k(\varepsilon)}$ and $\{v^k\}_{k \geq k(\varepsilon)} \subset B(\bar{v}, \delta(\varepsilon))$. This implies that for all $k \geq k(\varepsilon)$,

$$\max_{j=1,\ldots,t_Y+1} \lambda(v^k, x(k)) \leq \varepsilon\Lambda \leq \mu_k. \tag{32}$$

Then, for any $k \geq k(\varepsilon)$, an optimal solution $x(k)$ of problem $(SP1(v^k, \mu_k))$ solves problem $(SP(v^k))$ (Bazaraa, Sherali and Shetty [1], Chapter 9, Theorem 9.3.1). Hence, $\theta_{v^k}(x(k)) = 0$ for all $k \geq k(\varepsilon)$. The proof is complete. □

6 Conclusion

In this paper, we show that there exists an exact penalty parameter of an inner approximation method proposed by Yamada, Tanino and Inuiguchi [9]. This implies that for a sufficiently large penalty parameter $\mu_1 > 0$, an optimal solution of the reverse convex programming problem can be obtained by executing the inner approximation algorithm using penalty functions without replacing a penalty parameter $\mu_k > 0$.

References

[1] Bazaraa, M.S., H.D. Sherali and C.M. Shetty, *Nonlinear Programming: Theory and Algorithms*, 2nd ed., John Wiley, New York (1993).

[2] Hestenes, M.R., *Optimization Theory: The Finite Dimensional Case*, John Wiley, New York (1975).

[3] Hogan, W.W., "Point-to-Set Maps in Mathematical Programming", *SIAM Review*, Vol. 15, No. 3 (1973), pp.591−603.

[4] Horst, R. and H. Tuy, *Global Optimization: Deterministic Approaches, Third, Revised and Enlarged Edition*, Springer-Verlag, Berlin (1996).

[5] Horst, R. and P.M. Pardalos, *Handbook of Global Optimization*, Kluwer Academic Publishers, Dordrecht (1995).

[6] Konno, H., P.T. Thach and H. Tuy, *Optimization on Low Rank Nonconvex Structures* Kluwer Academic Publishers, Dordrecht (1997).

[7] Rockafellar, R.T., *Convex Analysis* Princeton University Press, Princeton, N.J. (1970).

[8] Tuy, H., *Convex Analysis and Global Optimization*, Kluwer Academic Publishers, Dordrecht (1998).

[9] Yamada, S., T. Tanino and M.Inuiguchi, "Inner Approximation Method for a Reverse Convex Programming Problem", *Journal of Optimization Theory and Applications*, Vol. 107, No. 2 (2000), pp.357−391.

逆凸計画問題に対する内部近似法の正確なペナルティパラメータ

山田修司　谷野哲三　乾口雅弘

概要

本研究では，コンパクトな凸集合の補集合を制約に持つ逆凸計画問題について考察する。集合 X が凸多面体でない場合の逆凸計画問題に対して，内部近似法に基づく逐次近似解法が Yamada, Tanino and Inuiguchi [9] によって提案されている。本研究では，そのアルゴリズムに対して正確なペナルティパラメータの存在性を示す。正確なペナルティパラメータの存在性より，アルゴリズムの各反復において，ペナルティパラメータを更新せずに逆凸計画問題の最適解が得られることがわかる。